NYO-1480-184
MF-69

Courant Institute of
Mathematical Sciences

Magneto-Fluid Dynamics Division

Classical Plasma Diffusion

Harold Grad

AEC Research and Development Report

Plasma Physics
November 1971

NYO-1480-184
c. 2

New York University

NEW YORK UNIVERSITY
COURANT INSTITUTE-LIBRARY

UNCLASSIFIED

Magneto-Fluid Dynamics Division
Courant Institute of Mathematical Sciences
New York University

MF-69 NYO-1480-184

CLASSICAL PLASMA DIFFUSION

Harold Grad

November 1971

Presented at Conference on Plasma Theory,
Kiev, October 19-23, 1971

U. S. Atomic Energy Commission
Contract No. AT(30-1)-1480

UNCLASSIFIED

Abstract

Macroscopic plasma diffusion as governed by the simplest form of Ohm's law for small mean free path is a very complex phenomenon which is by no means described by the classical "$1/B^2$" formula. Diffusion across a magnetic field is, in general, a nonlocal, nonlinear, and transient combination of skin effect, field diffusion, plasma diffusion, and singular convection. It turns out that several orders of magnitude are at stake in balancing these competing effects. There is probably at least as large an unknown factor to be determined in the equivalent calculation still to be done with a kinetic physical model. The basic question, when can transport coefficients be calculated as averages over flux surfaces, is examined.

1. Introduction

We consider a macroscopic resistive model subject to Ohm's law in the form

$$E + u \times B = \eta J \ . \qquad (1.1)$$

The value of η is left open; it can be classical or "anomalous" (and some predictions turn out to be independent of the value of η[1,2,3]). To complete the system, we can take conservation of mass, momentum, and flux in the form

$$\frac{\partial p}{\partial t} + \mathrm{div}(pu) = 0 \qquad (1.2)$$

$$\nabla p = J \times B \qquad (1.3)$$

$$\frac{\partial B}{\partial t} + \mathrm{curl}\ E = 0 \ . \qquad (1.4)$$

The isothermal assumption and absence of mass source implicit in (1.2), also the scalar pressure in (1.3) and scalar resistivity in (1.1) are easily generalized. But, in order to decouple diffusion from MHD wave motion, the inertia term, $\rho du/dt$, missing from (1.3), must remain excluded.

This is a much simpler physical model than the ones treated in most of the recent literature which couple energy flow with plasma diffusion, usually in a kinetic guiding center or drift model (e.g. [4,5]). But it was only recently recognized [1] that the standard calculation of diffusion in even the simplest macroscopic version (1.1) is woefully inadequate. It is based on a simplification, viz. curl E = 0 (the special theory), which turns out to eliminate from consideration all but a very special class of plasma equilibria. It can be shown [1,2,3] that this special class of equilibria sometimes takes over after a sufficient lapse of time (probably too long to be relevant in present experimental Tokomaks), but more often the special theory is not representative of the general case. Independent of the quantitative relevance of the macroscopic model (1.1) to a hot plasma with large mean free path, its qualitative understanding is crucial to the understanding of kinetic models for which the theory is still at the earlier stage of the standard macroscopic calculation with curl E = 0. The neoclassical orbit considerations supply some of the physics which is missing in the macroscopic model; but the additional physics which is missing because of the specialized profiles in the kinetic model is likely to yield equally important quantitative corrections.

The basic reason for the complexity of classical diffusion (also energy flow) is that what is called a "diffusion (heat transfer) coefficient" is not a traditional transport coefficient. It is not a local property of the plasma; rather it is a "black box" solution of a global boundary value problem. We can compare the elementary concept of a local viscosity coefficient with the drag coefficient for an airfoil, involving global interactions of wakes, boundary layers, shock waves, etc. -- each one dependent on viscosity; plasma diffusion is more akin to the drag coefficient than to the viscosity coefficient. The special theory (curl E = 0) can frequently be reduced to a calculation of local transport coefficients; the general theory almost never can.

This distinction is not mere semantics. For example, all practical numerical calculations of Tokomak behavior are done in a straight, cylindrical geometry with a diffusion (or heat transfer) coefficient which (among other ad hoc factors) is theoretically corrected for the toroidal curvature (e.g. [6,7]). But the correction factor is taken from a global, geometry-sensitive, convective calculation (such as [4] or [5]); moreover the theoretical calculation is valid only after complete relaxation of a transient -- and it is the transient which is the object of study of the numerical calculation!

Without inertia in (1.3), at least two time scales appear, separated by three or more orders of magnitude. With inertia, faster MHD time scales appear, increasing the spread by at least two more orders of magnitude. No numerical calculation which includes the fastest scale (MHD waves and compression) can include the slowest. But elimination of inertia makes the system very nonstandard with constraints replacing time derivatives; thus elaborate theoretical preparation is required before one can proceed with a numerical calculation [3].

Basic features of the general theory were described in [1]; some specific Tokomak applications have been given in [2]; details of the general theory will appear [3]. In this paper we briefly summarize these results and discuss the overall significance of constraints and averaging on transport properties.

2. Critique of the Special Calculation

Almost all diffusion calculations (both macroscopic and kinetic) are based on two postulates:

(i) small resistivity, implying slow passage through a family of quasistatic equilibria ($\nabla p = J \times B$ or an equivalent guiding center equilibrium).

(ii) low β, implying curl $E = 0$.

Although these seem eminently plausible, after examination they turn out to be unduly restrictive.

There are several consequences of these assumptions, of which we list three:

(a) the net flow can be calculated explicitly and very easily

(b) the pertinent equilibria are restricted

(c) the conclusions are very sensitive to slight changes in the model.

Implicit in postulate (i) above is that there *exists* a family of neighboring equilibria (through which the plasma slowly diffuses). This is not necessarily so because of the phenomenon of *marginal equilibrium* [8]. For example, a mirror machine which is bent into a bumpy torus may require an abrupt change in the topology of the pressure surfaces (this phenomenon is distinct from the formation of *magnetic* islands by loss of symmetry). Similarly, in a straight circular screw pinch, certain helical perturbations *must* change the pressure topology. In addition to its effect on diffusion, this also requires reexamination of stability concepts such as resistive tearing "modes". Standard estimates of resistive growth rates are irrelevant since they do not describe an instability at all but a change in equilibrium topology; to follow the consequences of the change in topology requires a nonlinear calculation of an initial value problem rather than normal modes or growth rates.

With regard to point (a), the calculation is miraculously able to ignore all the fine details of the global interplay between convection and diffusion, and it can always be carried out, both when it is correct and incorrect. For example, it is known that neither scalar pressure nor guiding center equilibria exist in asymmetric geometries [9]; but the calculation, nevertheless, yields a diffusion rate [10,11]; (this is not an abstract objection -- it is easily verified that the "solutions" do not satisfy the equations).

With regard to consequence (b) above, when there is no driving emf the mean parallel current on each magnetic surface must vanish [12]

$$\langle \eta J \cdot B \rangle = 0 \tag{2.1}$$

More generally [3], in a Tokomak or multipole geometry with driving emf's, the restriction

$$\langle \eta J \cdot B \rangle = c_1 \psi_1'(V) + c_2 \psi_2'(V) \tag{2.2}$$

must be satisfied where

$$c_i = \oint_{C_i} E \cdot dx \tag{2.3}$$

are the two (poloidal and toroidal) constant, path-independent, emf's, and $\psi_1 = \psi$ and ψ_2 are the poloidal and toroidal fluxes.

The general static equilibrium is characterized by two arbitrary profiles, e.g. $p(V)$ and $\psi(V)$, or $p(\psi)$ and $q(\psi) = \psi_2'(\psi)$ (q is the reciprocal rotation number or Tokomak "stability" parameter). The standard calculation, based on assumptions (i) and (ii), is automatically limited to a *single* arbitrary profile through the conditions (2.1) or (2.2). A natural question is whether these special profiles are representative of the general case. The answer is that the evolution in time (including diffusion properties)is even *qualitatively* different for the special and general profiles, both physically and analytically. The analytical features have necessitated development of an elaborate and entirely nonstandard mathematical structure [3]. As just one example, the classical boundary condition that at a fluid interface the normal component of fluid velocity equals the velocity of the interface is violated in this theory.

Some of the physical conclusions arising in the general theory which may be more immediately relevant to Tokomak experiments have been summarized in Ref. [2]:

(1) present Tokomaks are inherently transient since they do not persist long enough to settle down; this casts some doubt on the significance of scaling laws.

(2) the fact that the plasma does not *appear* to be transient is probably due to the dominant effect of the neutral influx.

(3) noncircular plasma cross-section can give a decided

advantage with regard to diffusion rate and β (a stability advantage has been noted previously [13]).

This transient conclusion was seemingly disputed in a later calculation [14]. There is actually no contradiction [15], since a steady state is assumed in Ref. 14 which is an ad hoc calculation based on several experimental inputs which would undoubtedly change if the experiment were extended sufficiently in time.

With regard to point (c), the sensitivity of the standard calculation, we point to at least three distinct sources. The classical skin effect has a diffusion coefficient $D_o \sim \eta/\mu_o$. The "classical" plasma diffusion coefficient is approximately $D_1 \sim \beta D_o$. Diffusion in a general equilibrium is characterized by a competition between these two processes. Since $\beta \sim 10^{-3}$ in a Tokomak and possibly 10^{-8} in an alkali plasma, even a very slight coupling with the much faster skin effect might raise D_1 enormously; (it is surprising that only factors of one or two orders of magnitude have so far arisen in the theory).

Another cause of great sensitivity is the complexity of the convective flow. Comparing the two different weighted averages

$$\begin{gathered} \langle u \cdot \nabla\psi \rangle \sim \text{net flow} \\ \langle r^2 \rangle \langle u \cdot \nabla\psi / r^2 \rangle , \end{gathered} \tag{2.4}$$

we should expect them to be approximately equal in a large aspect torus ($\delta r/r$ is small on a flux surface), but instead we find that their ratio is a large number ($\sim q^2$ where $2.5 < q < 4$ in a Tokomak). In a finite aspect torus, where the two expressions can be expected to be somewhat different, they are even found to scale differently. The point is that convection is strongly geometry dependent (whereas a true transport coefficient would not be).

Related to this phenomenon is that fact that D_1 for a straight circular cylinder is increased by a factor $1 + 2q^2$ (Pfirsch-Schluter [16])in a torus of arbitrarily large aspect ratio even though q itself is given the same value in the straight and toroidal geometries.

A similar calculation with resistivity $\eta \neq \eta(\psi)$ varying slightly (i.e. of order 1/aspect) on a flux surface gives a correction factor to D_1 which may be large compared to unity [3,15] just as for the Pfirsch-Schluter factor.

Another independent cancellation occurs in a Tokomak (or in any discharge with $\oint E \cdot dx \neq 0$). In a passive plasma, the net diffusion is directly related to the dissipation $\langle \eta J^2 \rangle$; in a discharge it is smaller by an order of magnitude. A similar sensitivity to the presence of a driving emf has been found for large mean free path in the form of trapped particle pinching [17].

3. The General Classical Calculation

With a general static equilibrium profile, the assumption curl E = 0 is not justified even with arbitrarily small resistivity and β. One contributive factor is that the skin effect (field penetrating plasma) and plasma diffusion (plasma penetrating field) are coupled through the pressure balance, giving rise to a complex convective flow which modifies both primitive concepts. The skin effect in a deformable plasma is not the same as in a resistive solid conductor, since this would clearly violate the pressure balance. The plasma diffusion which accompanies the skin penetration is entirely nonstandard, even in scaling. There is a very strong geometrical dependence; circular sections are qualitatively different from noncircular; Tokomaks are qualitatively different from hard cores or multipoles; there are singular limits as the toroidal or poloidal field dominates the other; the skin effect can involve one field component or both, and in a θ-pinch or bumpy torus there is no skin effect at all on the scale D_o. Not only the physical conclusions, but the mathematical structures differ in these cases. Entirely different numerical schemes are required in order to "march" in different cases, and a physical variable (such as ψ) which is advanced because of a time derivative in one case ($\partial\psi/\partial t$) will advance in virtue of an implicit constraint in another case.

In a large number of cases a standard sequence, skin effect followed by "classical" diffusion can be verified theoretically. For example, in a low β, large aspect Tokomak, the toroidal current will decay quickly in a time related to

D_o. This process will approximate the standard skin effect if the plasma cross-section is approximately circular; it will have the same scaling but will differ quantitatively from the standard skin effect if noncircular. During this process the pressure contours follow the changes in flux surface shape; the pressure remains approximately constant on a contour of fixed volume. The toroidal field cannot be said to diffuse, instead it fills in the gaps to maintain pressure balance. During the skin diffusion the plasma does diffuse slightly relative to a fixed volume, but at a nonstandard rate. The diffusion rate is not local and the process cannot be described in terms of mean values. After a number of skin times, the system settles down to one of the special profiles of the standard calculation. Provided that certain additional conditions are met, the system then diffuses slowly to a universal equilibrium with no adjustable parameters, independent of the initial profile (and even, to some extent, independent of the value of η [2]).

This standard sequence is not necessarily followed, e.g. at finite β or even at low β at finite aspect. The ultimate equilibrium may not be reached because of instability (this is a new type of resistive instability), or it may not be reached because it does not exist. For example, a Tokomak equilibrium in a straight circular cylinder can be found with a suitable mass source (ionization of neutrals) taking the place of the discharge, i.e. with $\oint E \cdot dx = 0$ [2]. But no such equilibrium exists for arbitrarily small toroidal curvature [3]. This is significant with regard to the "bootstrap Tokomak" concept [18,19].

By varying the plasma cross-section, the rate of approach to the universal equilibrium can be accelerated, and the plasma can be given more interesting parameters (e.g. higher β) [2].

A beginning of a transient (i.e. of a general) theory can be found in the guiding center model as the "trapped particle pinch effect" [17].

The details of the diffusion theory for a general profile are quite elaborate [3]; one qualitative reason for the vast difference between the special and the general theory can be described in terms of their relation to irreversible thermodynamics, which we now discuss.

4. Transport Coefficients, Symmetry and Averaging

Perhaps one reason that the general theory of diffusion for the classical system (1.1)-(1.4) remained undeveloped for so long is that it cannot be put into a standard "transport coefficient" framework as can the special theory. It turns out, however, that even the special theory becomes standard only with further restrictions. This is best seen after we discuss several simpler examples.

All standard transport coefficients are subject to the Onsager symmetry relations. However, there are additional nonthermodynamic symmetries which arise from quite a different source, as for example through a Green's function or topological symmetry. Entirely new problems arise when forces or fluxes are made subject to constraints or are averaged. Also, the extension of the symmetry principle to include skew symmetry for terms linear in B (Hall effect) may not hold for global or averaged quantities.

Consider first the classical problem of the flow of heat in a simply connected solid body (Fig. 1). We single out portions of the boundary, $S_1, S_2, \ldots$ which are held at temperatures $T_1, T_2, \ldots$ Solutions of the elliptic system

$$q = -\lambda \nabla T, \quad \operatorname{div} q = 0 \tag{4.1}$$

can be described by a symmetric Green's function. Thus the net heat flow

$$Q_i = \int_{S_i} q \cdot dS \tag{4.2}$$

is related to the set of temperatures T_i by a symmetric heat conductivity matrix

$$Q_i = \Lambda_{ij} T_j \tag{4.3}$$

The original problem has a single local transport coefficient λ and consequently has no Onsager identities. The discrete formulation in terms of T_i and Q_i gives an array of forces and fluxes, and the symmetric relation (4.3) arises from the symmetry of the Green's function.

Consider next the dissipative relation

$$E = \eta J \tag{4.4}$$

in a hollow toroidal copper shell (Fig. 2). We suppose that $\eta(x)$ is given and that

$$\operatorname{curl} E = 0, \quad \operatorname{div} J = 0 \tag{4.5}$$

and $J_n = 0$ on the boundary. This is a classical elliptic system, and the manifold of solutions is known to be a two-dimensional linear vector space [20]. A unique solution is determined by specifying a pair of periods, either emf's

$$\oint_{C_i} E \cdot dx = c_i \tag{4.6}$$

or currents

$$\int_{S_i} J \cdot dS = I_i \tag{4.7}$$

(see Fig. 2), which are related by the symmetric resistance matrix,

$$c_i = R_{ij} I_j \tag{4.8}$$

As in the previous example, a single local transport coefficient $\eta(x)$ proliferates to an array of forces and fluxes when examined globally.

In an abstract formulation,

$$v_i = A_{ij} u_j \tag{4.9}$$

where v_i are the "fluxes", u_j the "forces", and A_{ij} is any nonsingular matrix, we can symmetrize trivially by choosing appropriate new fluxes $\bar{v}_i$. Given any symmetric nonsingular L_{ij}, we simply set

$$\bar{v}_i = L_{ij} u_j \tag{4.10}$$

where the new fluxes $\bar{v}$ are defined in terms of the old by $\bar{v} = LA^{-1} v$ (there are only minor modifications necessary if the forces and fluxes are not independent and A and L are singular). Thus one can always symmetrize. The problem is whether this occurs through a natural or evident choice of variables in a given physical problem.

The proper abstract formulation of Irreversible Thermodynamics is to take forces and fluxes as dual linear vector spaces (in the standard terminology of linear algebra) with the entropy production as the defining inner product. The question, whether an abstract inner product (u,v) is symmetric has no meaning. The symmetry of a specific matrix representation of (u,v) depends on the choice of base vectors in u and in v. Similarly, we have just seen that an abstract mapping from u to v, such as represented by (4.9) or (4.10), can be symmetric or not depending on the choice of base vectors. However, the symmetry of a map from u to v relative to a given inner product (u,v) does have an invariant meaning. This is the content of the Onsager principle, that the relation between forces and fluxes is symmetric relative to a specific bilinear form, the entropy production.

If in the examples above the medium is anisotropic and there is a matrix relating the three components of J and E, or if there are several coupled vector fluxes J^{α} and forces E^{α}, then we may have local Onsager identities at a given point x of the medium. The global symmetry, relating emf to total current, is a consequence of two distinct symmetries, one thermodynamic and local, and the other primarily topological.

The geometrical symmetry can be separated out and is also related to an inner product. In a domain of arbitrary topology, the linear vector space of closed curves C_i is isomorphic to the linear vector space of cross cuts S_i (Alexander duality theorem, cf [20]). A reciprocal basis can be found so that

$$(C_i, S_j) = \delta_{ij} \tag{4.11}$$

where (C_i, S_j) is the algebraic number of intersections of the curve C_i with the surface S_j.

Given two vector fields X and Y satisfying

$$\begin{aligned} \operatorname{curl} X &= 0 \\ \operatorname{div} Y &= 0 \end{aligned} \tag{4.12}$$

in the domain and $Y_n = 0$ on the boundary, an easy calculation gives

$$(X,Y) = \int X \cdot Y \, dV = \sum_1^n \xi_i \eta^i \tag{4.13}$$

where

$$\xi_i = \oint_{C_i} X \cdot dx, \qquad \eta^i = \int_{S_i} Y \cdot dS \tag{4.14}$$

(many generalizations and extensions of this formula, Hodges' Theorem, are given in [20]). The system (4.12) is incomplete and does not determine X or Y. To complete the system we need a linear relation between X and Y. The simplest, X = Y, determines a manifold of solutions which is an n-dimensional linear vector space spanned by either ξ_i or η^i. If Z is a specific solution, its dual representations

$$Z = \Sigma \xi_i X^i = \Sigma \eta^i Y_i \tag{4.15}$$

where X^i and Y_i are reciprocal bases relative to the inner product (4.13),

$$(X^i, Y_j) = \delta^i_j \tag{4.16}$$

gives the symmetric matrix

$$\xi_i = (Y_i, Y_j)\eta^j \tag{4.17}$$

Taking $\xi_i = c_i$ and $\eta_j = I_j$, we have proved that $R_{ij} = (Y_i, Y_j)$ is symmetric.

Any local symmetric linear relation between X and Y which is used to complete the system will combine with the topological symmetry to give a globally symmetric result.

In a simple domain, dropping the boundary condition $Y_n = 0$ and setting $X = \nabla\Phi$, we have

$$(X, Y) = -\int \Phi Y_n \, dS \tag{4.18}$$

This symmetric inner product between boundary values of Φ and Y_n [which is the source of the symmetry (4.3)], coupled with a symmetric local (Onsager) relation between Y and $\nabla\Phi$ gives a globally symmetric structure.

To illustrate the effect of averaging, consider an abstract mathematical model of forces $u_i(x)$ and fluxes $v_i(x)$ related by a symmetric matrix $L_{ij}(x)$

$$v_i(x) = L_{ij}(x)\, u_j(x) \tag{4.19}$$

If u_j = const. (independent of x), then averaging clearly preserves the symmetric structure,

$$\int v_i dx = \left(\int L_{ij} dx\right) u_j \tag{4.20}$$

The same is true if v_i = const. in terms of the reciprocal matrix, L^{-1}. The conditions u_i = const. or v_i = const. are natural and not arbitrary. For example, minimizing the quadratic form

$$(u, v) = \int v_i L^{-1}_{ij} v_j dx \tag{4.21}$$

subject to fixed total flux

$$\int v_j dx = V_j \tag{4.22}$$

gives, by the Lagrange multiplier rule,

$$\delta\{\int v_i L^{-1}_{ij} v_j dx - \lambda_i \int v_j dx\} = 0 \tag{4.23}$$

the natural minimization condition

$$2u_i = \lambda_i = \text{const.} \tag{4.24}$$

It is easy to see that there is no natural averaging which preserves the linear structure when both forces and fluxes depend on x.

We can now understand the successful averaging in the first two examples given above. The heat flow problem (4.1) gives a simple symmetric averaged structure (4.3) when the T_i are taken to be constant on each boundary segment; otherwise it does not. Similarly, the electrical conduction problem (4.4), (4.5) gives a simple averaged result because curl E = 0 guarantees that the forces c_i are constant (path-independent).

Averaging occurs in a more complex way if J is contrained to lie in given surfaces, or if, because of highly anisotropic heat conductivity, T is constrained to be constant on specified surfaces. The standard transport formulation must be modified when there is a constraint and, as we might expect, the inner product gives the clue.

If T is constrained to be constant on a given set of surfaces, $T = T(\psi)$, the inner product is

$$\int \nabla T \cdot q \, dV = \oint T \, q \cdot dS - \int T \, \mathrm{div} \, q \, dV \qquad (4.25)$$

With $T = T(\psi)$, the volume term vanishes if $\langle \mathrm{div} \, q \rangle = 0$ on each ψ-surface. Thus restricting the admissible T forces us to expand the admissible q from div q = 0 at each point to $\langle \mathrm{div} \, q \rangle = 0$ on each surface.

If J is contrained to lie in a specified surface, $\nabla\psi \cdot J = 0$, we can write

$$J = \nabla\psi \times \nabla\omega \qquad (4.26)$$

and evaluate

$$\begin{aligned}\int E\cdot J\,dV &= -\int E\cdot \mathrm{curl}(\omega\nabla\psi)\,dV \\ &= -\int \omega(\nabla\psi\cdot \mathrm{curl}\ E)\,dV + \text{boundary terms}\end{aligned} \qquad (4.27)$$

Thus the restriction on J requires us to relax E from curl $E = 0$ to $\nabla\psi\cdot$curl $E = 0$; E is a surface gradient on each ψ-surface. Note that taking the conventional equations div $q = 0$ or curl $E = 0$ is incorrect (and overdetermines the system) when there is a constraint such as $T = T(\psi)$ or $\nabla\psi\cdot J = 0$. (In the plasma diffusion problem we are permitted to retain curl $E = 0$ rather than $\nabla\psi\cdot$curl $E = 0$ only because Ohm's law is relaxed to include an adjustable term $u \times B$.)

The constrained current problem is formulated as

$$\begin{aligned} E = \eta J, &\quad \nabla\psi\cdot \mathrm{curl}\ E = 0 \\ \mathrm{div}\ J = 0, &\quad \nabla\psi\cdot J = 0 \end{aligned} \qquad (4.28)$$

On each surface we have an elliptic system and (4.8) generalizes to

$$c_i(\psi) = R_{ij}(\psi)I_j'(\psi) \qquad (4.29)$$

where $I_j(\psi)$ is the total current within the ψ-surface. A simple symmetric structure obtains on each ψ-surface since the forces c_i are constant on a surface. But no such structure exists when averaged over ψ unless curl $E = 0$ rather than $\nabla\psi\cdot$curl $E = 0$.

Up to now we have taken a restricted Onsager principle without a magnetic field. With Hall effect, the relation between forces and fluxes is no longer symmetric. It is conventional to include the skew symmetry of terms linear in B (this is not a physical principle at all, but a consequence of the fact that B is not a vector) as an extension of the "symmetry" principle. This convention is not tenable when we turn to global considerations. A Hall term in Ohm's law, $E = \eta J + \sigma J \times B$ or in heat conduction, $q = -\lambda \nabla T + \tau q \times B$, makes the differential system non-self adjoint, destroys the symmetry of the Green's function, and usually destroys the global symmetry between forces and fluxes. Symmetry cannot be restored by any expedient related to reversal of part of the matrix, since, for example, various magnitudes and orientations of B may be averaged globally.

The simplest example with Hall effect is two-dimensional heat flow; we take the same problem as before, Fig. 1. We have

$$q = -\lambda \nabla T + \tau q \times B \tag{4.30}$$

where B is a constant vector perpendicular to the plane of interest, and λ and τ are constant. Using div $q = 0$, we can set

$$q = n \times \nabla \psi \tag{4.31}$$

where n is the transverse unit vector. We have

$$n \times \nabla\psi = \nabla\Phi \tag{4.32}$$

where

$$\Phi = \tau B\psi - \lambda T \tag{4.33}$$

From (4.32), we see that Φ and ψ are conjugate harmonic potentials related by a symmetric Green's function. In particular, Φ = const. on each segment S_i of the boundary gives a symmetric global relation between Φ_i and Q_i. But Φ = const. on S_i is inconsistent with T = const. (ψ must vary if there is any heat flow through S_i). In fact, we have symmetrized by changing bases in (4.33)(as we have shown, this can always be done). Even without making the problem discrete, i.e. without averaging, we can verify that the boundary values $q_n(S)$ are not symmetric in terms of T(S), although they are symmetric in terms of $\Phi(S)$.

It should be pointed out that when a Hall term such as $q \times B$ points in an innocuous direction, the desymmetrizing effect may disappear. For example, take

$$E = \eta J + \sigma J \times B \tag{4.34}$$

together with the remainder of (4.28)(note that B is a given constraining field with $B\cdot\nabla\psi = 0$, and in this context, has no relation to J). The surface equation is the same as before, and the normal component of (4.34) merely serves to determine the component $E\cdot\nabla\psi$ after J has been found. Note that this system is not self adjoint, yet (4.29) remains symmetric.

More generally, a set of fluxes q^{α} and forces $\nabla T^{\alpha}(\psi)$ gives a symmetric averaged matrix even with Hall effect. The same is true of a set of fluxes J^{α} each constrained by $\nabla\psi\cdot J^{\alpha} = 0$. However, a system with mixed constraints, coupling $(q,T), T = T(\psi)$ with $(J,E), J\cdot\nabla\psi = 0$, is not symmetric and does not allow averaging in the presence of Hall effect.

Turning to the plasma diffusion problem, since there is only one local transport coefficient, η, any global symmetries that arise are not basically thermodynamic. With the special equilibrium (2.2), the dissipation takes the form

$$\int \eta J^2 dV = c_1 I_1 + c_2 I_2 - \int (u\cdot\nabla p)\, dV \qquad (4.35)$$

where I_1 and I_2 are the total poloidal and toroidal currents. An unusual, symmetric "matrix" arises [3] relating the forces

$$c_1, c_2, -p'(V) \qquad (4.36)$$

and the fluxes

$$I_1, I_2, \oint_V u\cdot dS \qquad (4.37)$$

Note that c_1, c_2 and I_1, I_2 are global (averaged over the entire domain), whereas $p'(V)$ and $\oint u\cdot dS$ are local to a surface. The symmetry under averaging is preserved because c_1 and c_2 are constant (curl $E = 0$) and p is constant on a flux surface. Variation of η with ψ or even over a flux surface is irrelevant to the structure (but can change the numerical values considerably).

It is immediately evident that inclusion of heat flow, either by itself or coupled to diffusion, can yield a standard structure after averaging over flux surfaces only when $T = T(\psi)$. This restriction is inadmissible in the present theory, but it may be valid at very large mean free path. Coupled heat flow and diffusion yields an innocuous Hall effect (both are of type q^α above). But in a discharge, $\oint E \cdot dx \neq 0$, symmetry is lost and there is no longer any simple relation between averaged forces and fluxes (since the system now has mixed constraints).

Similarly, with fluid flow (steady state rather than static equilibrium) and Hall effect, since the nondissipative equilibrium has distinct flux and pressure surfaces, $p \neq p(\psi)$ [21], even diffusion by itself cannot be described in terms of net flow and other flux surface averages. In a guiding center model, the equilibrium can sometimes be written [22]

$$K \times B = (\partial p/\partial \psi)\nabla\psi$$

$$K = \mathrm{curl}\ (\sigma B) \tag{4.38}$$

$$\sigma = 1 - (\partial p/\partial B)/B$$

where $p = p(\psi, B)$ is the parallel pressure. K takes the place of J and is constrained to lie in ψ-surfaces. In Ohm's law,

$$E + u \times B = \ldots \qquad (4.39)$$

multiplication by K gives an averaged term $\langle E \cdot K \rangle$ which can be evaluated in terms of global emf's, but

$$\langle u \times B \cdot K \rangle = - \langle (u \cdot \nabla\psi) \partial p / \partial\psi \rangle \qquad (4.40)$$

is not simply interpretable. Specializing to

$$p = p_1(\psi) p_2(B) \ , \qquad (4.41)$$

we have

$$\langle u \times B \cdot K \rangle = -p_1' \langle (u \cdot \nabla\psi) p_2 \rangle \qquad (4.42)$$

which indicates that the force p_1' may be associated with a weighted plasma flow. The net plasma flow occurs in an averaged calculation only to the extent that B = const. on flux surfaces (see also [15]).

In a kinetic model, there are no local transport coefficients; it requires flux surface averaging to introduce this concept at all. There is the same distinction between a general and a special theory (the latter with curl E = 0 and restricted profiles). As in the small mean free path theory, there are no averaged transport coefficients in the general theory, and there are strict limitations on the use of averaging in the special theory. The consequence is that there is no widely applicable theory of transport coefficients, either locally or averaged over surfaces. In other words, each problem is solved globally as a special boundary value problem for a special profile, and there is no direct carry-over to another problem with different shape or profile (as there would be for a legitimate transport coefficient).

To elucidate the general theory, consider the relation between conservation equations and dissipation. Conservation of energy in a static fluid can be written

$$C_V \frac{\partial T}{\partial t} + \text{div } q = 0 \tag{4.43}$$

which together with the transport relation

$$q = -\lambda \nabla T \tag{4.44}$$

gives a simple dissipative diffusion equation for T. Note that the force is a gradient of the quantity which is conserved. The isothermal conservation of mass

$$\frac{\partial p}{\partial t} + \text{div } (pu) = 0 \tag{4.45}$$

together with "classical" plasma diffusion,

$$u = -D_1 \nabla p/p \tag{4.46}$$

gives a similar diffusion equation for p. Conservation of flux,

$$\frac{\partial B}{\partial t} + \text{curl } E = 0 \tag{4.47}$$

together with the transport relation $E = \eta J$, gives a similar diffusion equation for flux,

$$\frac{\partial B}{\partial t} = -\mathrm{curl}[(\eta/\mu_o)\mathrm{curl}\ B] \ . \tag{4.48}$$

There is a complete analogy of flux, mass, and energy conservation. But, curiously, the analogy holds only if E is called the flux and J the force; also the heat conductivity λ takes the same position as electrical resistivity η, viz. as a diffusivity coefficient. This has a number of interesting consequences which we shall not pursue here.

In all these time dependent formulations the transport relation itself does not depend explicitly on time. This is the crucial difference, in the present context, between the special plasma diffusion theory and the general theory. In the latter, although the local transport relations (when they exist) are always time-independent, the averages over flux surfaces can never be separated from the global transient solution of the entire system. There is no doubt that the situation will be found to be the same in all finite or large mean free path models when the general time-dependent case is attempted. We conclude that the special solutions, with curl $E = 0$, give limited information about plasma transport behavior in general. And even within the special theory, only in certain well-defined special cases do the transport properties reduce to a calculation of flux surface averages.

5. Conclusion

In a small mean free path plasma, local transport coefficients exist at each point of the plasma. Transport coefficients in a finite mean free path system can only exist as a consequence of averaging over flux surfaces. The condition curl E = 0 does not follow from low β but from a specialization of profiles (and sometimes from aging the system). In the special theory, with curl E = 0, in addition to the fact that this postulate represents only a small part of the physical world, averaging leads to a conventional system of transport coefficients only after additional specialization. Even when there do exist averaged transport coefficients, they frequently cannot be found directly from an entropy production quadratic form because of the interference of Hall effect which may destroy the averaged symmetry.

The general theory, with curl E $\neq$ 0, exhibits many new dissipative properties and requires new analytic techniques. It turns out that numerical calculations of transients must be done in an actual toroidal geometry. Theoretical results have thus far been obtained only for the macroscopic case, and even here mostly for diffusion, uncoupled to energy flow. It has been predicted [1,2] that corrections of up to two orders of magnitude should be expected to arise when this theory is generalized to finite or large mean free path problems.

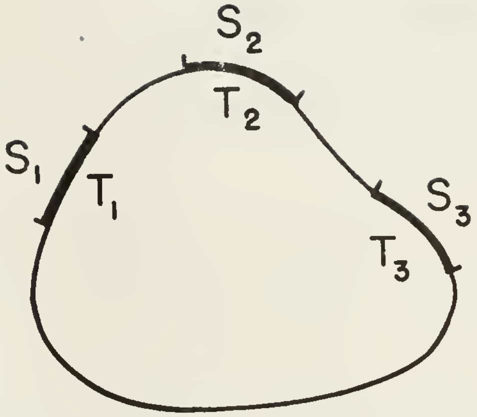

figure 1
Simply Connected Domain

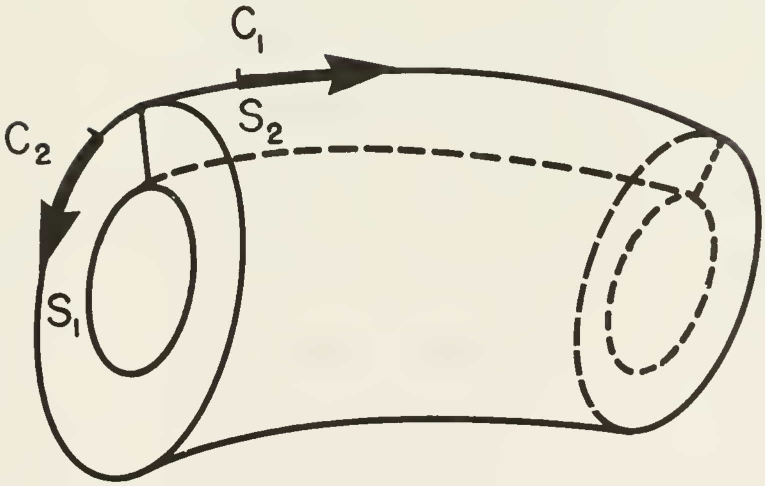

figure 2
Toroidal Domain

References

1. H. Grad and J. Hogan, Bull. Am. Phys. Soc., 14, 1017 (1969).

2. H. Grad and J. Hogan, Phys. Rev. Letters, 24, 1337 (1970).

3. H. Grad and J. Hogan, to appear.

4. A.A. Galeev and R.Z. Sagdeev, JETP, 26, 233 (1968) [Russian 53, 348 (1967)].

5. M.N. Rosenbluth, R. Hazeltine and F.L. Hinton, to appear.

6. Y.N. Dnestrovski, D.P. Kostamarov and N.L. Pavlova, Fourth European Conference on Plasma Physics, Rome 1970.

7. D.F. Düchs, H.P. Furth and P.H. Rutherford, "Radial Transport in Tokomak Discharges", IAEA Conference, Madison, Wisconsin, 1971, to appear.

8. H. Grad, "Plasma Containment in Closed Line Systems", IAEA Conference, Madison, Wisconsin, June 1971, to appear.

9. H. Grad, Phys. Fluids, 10, 137 (1967).

10. E. Frieman, Phys. Fluids, 13, 490 (1970).

11. J.L. Johnson and S. von Goeler, Phys. Fluids, 12, 255 (1969).

12. M.D. Kruskal and R.M. Kulsrud, Phys. Fluids, 1, 265 (1958).

13. T. Ohkawa and H.G. Voorhies, Phys. Rev. Letters, 22, 1275 (1969).

14. H.P. Furth et al., Phys. Fluids, 13, 3020 (1970).

15. J. Nührenberg, "Comparison Between Stationary Tokomak Models", to appear.

16. D. Pfirsch and A. Schluter, Max Planck Inst., Munich, Report No. MPI/PF/7/62 (1962).

17. A.A. Ware, Phys. Rev. Letters, 25, 916 (1970).

18. R.J. Bickerton, J.W. Connor and J.B. Taylor, Nature, 229, 110 (1971).

19. B.B. Kadomtsev and V.D. Shafranov, "Stationary Tokomak", IAEA Conference, Madison, Wisconsin, 1971 , to appear.

20. A.A. Blank, K.O. Friedrichs and H. Grad, Notes on MHD-V, NYO-6486, N.Y. University (1957).

21. D. Dobrott and H. Grad, "Steady Flow with Hall Effect", to appear.

22. H. Grad, Trans. N.Y. Acad. Sci. II, 33, 163 (1971).

MAR 1972 DATE DUE
AUG 1974
MAR 24 1976
GAYLORD PRINTED IN U.S.A.

Thi
Gov
Uni
per

A.

f
t,

ly

B.

om

As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor, to the extent that such employee or contractor of the Commission, or employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission, or his employment with such contractor.

NYO-1480-184 c.2

Grad

C

NYO-1480-184 **c.2**
Grad

NYO-1480-184 c.2
Grad

AUTHOR

Classical plasma diffusion.

TITLE

DATE DUE	BORROWER'S NAME
[illegible]	~~LAM~~
AUG - 6 [illegible]	
24 1976	

N.Y.U. Courant Institute of
Mathematical Sciences
251 Mercer St.
New York, N. Y. 10012

NEW YORK UNIVERSITY
COURANT INSTITUTE - LIBRARY